6.7.335. Use Stokes theorem to evaluate $\displaystyle\int_C \left[2xy^2z\,dx + 2x^2yz\,dy + (x^2y^2-2z)dz\right]$ where C is given by $x=\cos t, y=\sin t, z=\sin t, 0\le t\le 2\pi$, traversed in the direction of increasing t.

Solution.
At a first glance, it is considerably easier to straightforward calculate the integral over the curve, than to calculate curl, and calculate the flux of the curl over the surface. Unless there occurs a magical cancellation, making curl zero, or the product of the curl and the normal vector to the surface.
Let us calculate the curl.
Usually in texts the curl in formulas is written as $\nabla\times\mathbf{F}$, instead of $\operatorname{curl}\mathbf{F}$.
We have

$$\mathbf{F} = 2xy^2z\hat{\text{i}} + 2x^2yz\hat{\text{j}} + (x^2y^2-2z)\hat{\text{k}}$$

The curl is

$$\begin{aligned}\nabla\times\mathbf{F} &= \begin{vmatrix} \hat{\text{i}} & \hat{\text{j}} & \hat{\text{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xy^2z & 2x^2yz & (x^2y^2-2z) \end{vmatrix} \\ &= (2x^2y-2x^2y)\hat{\text{i}} + (2xy^2-2xy^2)\hat{\text{j}} + (2xy^2-2xy^2)\hat{\text{k}} \\ &= 0\end{aligned}$$

We are "lucky" - the curl is zero, so the flux of curl across surface is zero.

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = 0$$

6.7.344. Use Stokes theorem to evaluate line integral $\int_C (zdx+xdy+ydz)$, where C is a triangle with vertices $(3,0,0)$, $(0,0,2)$, and $(0,6,0)$ traversed in the given order.

Solution.
We have

$$\mathbf{F} = z\hat{\mathrm{i}} + x\hat{\mathrm{j}} + y\hat{\mathrm{k}}$$

$$\begin{aligned}\operatorname{curl}\mathbf{F} &= \begin{vmatrix} \hat{\mathrm{i}} & \hat{\mathrm{j}} & \hat{\mathrm{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z & x & y \end{vmatrix} \\ &= (1)\hat{\mathrm{i}} + (1)\hat{\mathrm{j}} + (1)\hat{\mathrm{k}}\end{aligned}$$

The Stokes theorem says

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$$

where S is a surface containing the curve C.
Note that S may be any surface containing C.
For us, the most convenient to choose the surface consisting from parts of coordinate planes:
$S_1 =$ triangle $(3,0,0),(0,0,2),(0,0,0)$, normal $\mathbf{N_1} = -\hat{\mathrm{j}}$, area $A_1 = \dfrac{3\cdot 2}{2} = 3$;
$S_2 =$ triangle $(0,0,2),(0,6,0),(0,0,0)$, normal $\mathbf{N_2} = -\hat{\mathrm{i}}$, area $A_2 = \dfrac{2\cdot 6}{2} = 6$;
$S_3 =$ triangle $(0,6,0),(3,0,0),(0,0,0)$, normal $\mathbf{N_3} = -\hat{\mathrm{k}}$, area $A_3 = \dfrac{6\cdot 3}{2} = 9$.
The minus signs are due to the orientation of the curve C (unless I am mistaken, and it should be all pluses).

Then

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} \\ &= \iint_{S_1} \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} + \iint_{S_2} \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} + \iint_{S_3} \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} \\ &= \iint_{S_2} (\hat{\mathrm{i}}+\hat{\mathrm{j}}+\hat{\mathrm{k}})\cdot(-\hat{\mathrm{j}})dS + \iint_{S_2} (\hat{\mathrm{i}}+\hat{\mathrm{j}}+\hat{\mathrm{k}})\cdot(-\hat{\mathrm{i}})dS + \iint_{S_3} (\hat{\mathrm{i}}+\hat{\mathrm{j}}+\hat{\mathrm{k}})\cdot(-\hat{\mathrm{k}})dS \\ &= -\iint_{S_1} dS - \iint_{S_2} dS - \iint_{S_3} dS \\ &= -3-6-9 \\ &= -18\end{aligned}$$

Why the surface consisting of 3 triangles S_1, S_2, S_3 is more convenient than simply one surface - a plane passing through the given triangle:
First, we would need to find the normal to the surface. Not very difficult, but some work:

The plane passing through the points $(3,0,0),(0,0,2)$, and $(0,6,0)$ is

$$\frac{x}{3}+\frac{y}{6}+\frac{z}{2}=1$$

The normal vector is

$$\mathbf{N}=\frac{1}{3}\hat{\mathrm{i}}+\frac{1}{6}\hat{\mathrm{j}}+\frac{1}{2}\hat{\mathrm{k}}$$

Then we need to normalize it.
And we need to find the area of S, which we would do by vector product.
Then to apply the formula, which will result in

$$\int_C \mathbf{F}\cdot d\mathbf{r}=\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}=(\operatorname{curl}\mathbf{F}\cdot\mathbf{S})\iint_S dS$$

6.7.355. Use Stokes theorem to compute $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x,y,z) = \hat{\imath} + xy^2\hat{\jmath} + xy^2\hat{k}$ and S is a part of plane $y+z=2$ inside cylinder $x^2+y^2=1$ and oriented counterclockwise.

Solution.
Here it is a reverse problem comparing to the previous two - instead of calculating the flux of curl of $\mathbf{F}$ through the surface, we should calculate a line integral of $\mathbf{F}$.

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$$

We need to express the line of intersection of the plane and the cylinder as a parametric function.
In x and y it is a circle:

$$x = \cos t, y = \sin t$$

Since $y+z=2$, it follows $z = 2-y = 2-\sin t$.

$$\underline{\mathrm{r}}(t) = \langle \cos t, \sin t, 2-\sin t\rangle$$
$$\mathbf{F}(\mathbf{r}(t) = \langle 1, xy^2, xy^2\rangle = \langle 1, \cos t\sin^2 t, \cos t \sin^2 t\rangle$$
$$\underline{\mathrm{r}}'(t) = \langle \cos t, \sin t, 2-\sin t\rangle' = \langle -\sin t, \cos t, -\cos t\rangle$$

$$\begin{aligned}
\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} &= \int_C \mathbf{F} \cdot d\mathbf{r} \\
&= \int_{t=0}^{2\pi} \langle 1, \cos t \sin^2 t, \cos t \sin^2 t\rangle \cdot \langle -\sin t, \cos t, -\cos t\rangle dt \\
&= \int_{t=0}^{2\pi} \left(-\sin t + \cos t \sin^2 t \cos t - \cos t \sin^2 t \cos t\right) dt \\
&= \int_{t=0}^{2\pi} (-\sin t)\, dt \\
&= 0
\end{aligned}$$

6.7.356. Use Stokes theorem to evaluate $\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$, where $\mathbf{F}(x,y,z) = -y^2\hat{\imath} + x\hat{\jmath} + z^2\hat{\mathrm{k}}$ and S is the part of plane $x+y+z=1$ in the positive octant and oriented counterclockwise $x \geq 0, y \geq 0, z \geq 0$.

Solution.
As in the previous prolem, we should calculate the line integral. The curve is the line of intersection of the plane $x+y+z=1$ with coordinate planes.
It is three lines:
C_1: $x+z=1, y=0$;
C_2: $x+y=1, z=0$;
C_3: $y+z=1, x=0$.
We can introduce a parameter t, but the simplest is to use already existing variables. The first line may be expressed parametrically with parameter x; x correctly increases as we go counterclockwise. Then $z=1-x$, $y=0$.

$$\mathbf{r}(x) = \langle x, 0, 1-x\rangle,\ \mathbf{r}'(x) = \langle 1, 0, -1\rangle, 0 \leq x \leq 1$$
$$\mathbf{F}(x) = -y^2\hat{\imath} + x\hat{\jmath} + z^2\hat{\mathrm{k}} = x\hat{\jmath} + (1-x)^2\hat{\mathrm{k}}$$
$$\begin{aligned}\int_{C_1} \mathbf{F}\cdot d\mathbf{r} &= \int_{x=0}^{1} \langle 0, x, (1-x)^2\rangle \cdot \langle 1, 0, -1\rangle dx \\ &= \int_{x=0}^{1} -(1-x)^2\, dx \\ &= -\frac{1}{3}\end{aligned}$$

The second line may be expressed parametrically with parameter y. (not x, because x decreases on this segment) Then $x = 1-y$, $z=0$.

$$\mathbf{r}(y) = \langle 1-y, y, 0\rangle,\ \mathbf{r}'(y) = \langle -1, 1, 0\rangle,\ 0 \leq y \leq 1$$
$$\mathbf{F}(y) = -y^2\hat{\imath} + x\hat{\jmath} + z^2\hat{\mathrm{k}} = -y^2\hat{\imath} + (1-y)\hat{\jmath}$$
$$\begin{aligned}\int_{C_2} \mathbf{F}\cdot d\mathbf{r} &= \int_{y=0}^{1} \langle -y^2, 1-y, 0\rangle \cdot \langle -1, 1, 0\rangle dy \\ &= \int_{y=0}^{1} (y^2+1-y)\, dy \\ &= \frac{1}{3} + 1 - \frac{1}{2} \\ &= \frac{5}{6}\end{aligned}$$

The third line may be expressed parametrically with parameter z. (not y, because y decreases on this segment) Then $y = 1-z$, $x = 0$.

$$\mathbf{r}(z) = \langle 0, 1-z, z\rangle,\ \mathbf{r}'(z) = \langle 0, -1, 1\rangle,\ 0 \leq z \leq 1$$
$$\mathbf{F}(z) = -y^2\hat{\imath} + x\hat{\jmath} + z^2\hat{\mathrm{k}} = -(1-z)^2\hat{\imath} + z^2\hat{\mathrm{k}}$$
$$\begin{aligned}\int_{C_3} \mathbf{F}\cdot d\mathbf{r} &= \int_{z=0}^{1} \langle -(1-z)^2, 0, z^2\rangle \cdot \langle 0, -1, 1\rangle dz \\ &= \int_{y=0}^{1} z^2\, dz \\ &= \frac{1}{3}\end{aligned}$$

Putting all three parts together,

$$\begin{aligned}\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} &= \int_C \mathbf{F}\cdot d\mathbf{r}\\ &= \int_{C_1}\mathbf{F}\cdot d\mathbf{r} + \int_{C_2}\mathbf{F}\cdot d\mathbf{r} + \int_{C_3}\mathbf{F}\cdot d\mathbf{r}\\ &= -\frac{1}{3}+\frac{5}{6}+\frac{1}{3}\\ &= \frac{5}{6}\end{aligned}$$

6.8.386. Use the divergence theorem to calculate surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = x^4\hat{\imath} - x^3z^2\hat{\jmath} + 4xy^2z\hat{k}$ and S is the surface bounded by cylinder $x^2+y^2 = 1$ and planes $z = x + 2$ and $z = 0$.

Solution.
The divergence theorem:

$$\iiint_E \operatorname{div} \mathbf{F}\, dV = \iint_S \mathbf{F} \cdot d\mathbf{S}.$$

Usually in texts the divergence in formulas is written as $\nabla \cdot \mathbf{F}$, instead of $\operatorname{div} \mathbf{F}$.

The divergence of $\mathbf{F}$ is

$$\begin{aligned}\nabla \cdot \mathbf{F} &= \nabla \cdot (x^4\hat{\imath} - x^3z^2\hat{\jmath} + 4xy^2z\hat{k}) \\ &= \frac{\partial}{\partial x}x^4 + \frac{\partial}{\partial y}(-x^3z^2) + \frac{\partial}{\partial z}(4xy^2z) \\ &= 4x^3 + 4xy^2\end{aligned}$$

The volume integral is most convenient in cylindrical coordinates.

$$x = r\cos\theta, y = r\sin\theta$$

z changes between $z = 0$ and $z = x + 2 = r\cos\theta + 2$.

$$\operatorname{div} \mathbf{F} = 4x^3+4xy^2 = 4(x^2+y^2)(x+y) = 4r^2(r\cos\theta+r\sin\theta) = 4r^3(\cos\theta+\sin\theta)$$

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iiint_E \operatorname{div} \mathbf{F}\, dV \\ &= \int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{z=0}^{r\cos\theta+2} 4r^3(\cos\theta + \sin\theta)r\, dz\, dr\, d\theta \\ &= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} (r\cos\theta + 2)4r^3(\cos\theta + \sin\theta)r\, dr\, d\theta \\ &= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} \left[4r^5\cos\theta(\cos\theta + \sin\theta) + 8r^4(\cos\theta + \sin\theta)\right] dr\, d\theta \\ &= \int_{\theta=0}^{2\pi} \left[\frac{2}{3}r^6\cos\theta(\cos\theta + \sin\theta) + \frac{8}{5}r^5(\cos\theta + \sin\theta)\right]_{r=0}^{1} d\theta \\ &= \int_{\theta=0}^{2\pi} \left[\frac{2}{3}\cos\theta(\cos\theta + \sin\theta) + \frac{8}{5}(\cos\theta + \sin\theta)\right] d\theta \\ &= \int_{\theta=0}^{2\pi} \left[\frac{2}{3}\cos^2\theta\right] d\theta \qquad \text{(the rest integrate to 0)} \\ &= (2\pi)(\frac{2}{3})(\frac{1}{2}) \\ &= \frac{2\pi}{3}\end{aligned}$$

6.8.390. Use the divergence theorem to compute the value of flux integral $\iint_S \mathbf{F}\cdot d\mathbf{S}$, where $\mathbf{F}(x,y,z) = (y^3+3x)\hat{\text{i}} + (xz+y)\hat{\text{j}} + [z+x^4\cos(x^2y)]\hat{\text{k}}$ and S is the area of the region bounded by $x^2+y^2=1, x\geq 0, y\geq 0$, and $0\leq z\leq 1$.

Solution.
The divergence theorem:

$$\iiint_E \operatorname{div}\mathbf{F}\, dV = \iint_S \mathbf{F}\cdot d\mathbf{S}.$$

$$\begin{aligned}\operatorname{div}\mathbf{F} &= \frac{\partial}{\partial x}(y^3+3x) + \frac{\partial}{\partial y}(xz+y) + \frac{\partial}{\partial z}[z+x^4\cos(x^2y)\\ &= 3+1+1\\ &= 5\end{aligned}$$

The region bounded by $x^2+y^2=1, x\geq 0, y\geq 0$, and $0\leq z\leq 1$, is a quarter of a cylinder of radius 1 and height 1;
$\operatorname{div}\mathbf{F}$ is constant.
This makes the solution easy.

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iiint_E \operatorname{div}\mathbf{F}\, dV\\ &= \iiint_E 5\, dV\\ &= 5\iiint_E dV\\ &= 5(\text{the volume of quarter-cylinder})\\ &= 5(\frac{\pi}{4}\cdot 1)\\ &= \frac{5}{4}\pi\end{aligned}$$

(The base is a quarter of a unit disk, so the area of the base is $\frac{\pi r^2}{4} = \frac{\pi}{4}$; the height is 1.)

6.8.402. Let $\mathbf{F}(x, y, z) = 2x\hat{\imath} - 3xy\hat{\jmath} + xz^2\hat{\mathrm{k}}$. Use the divergence theorem to calculate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where S is the surface of the cube with corners at $(0, 0, 0), (1, 0, 0), (0, 1, 0), (1, 1, 0), (0, 0, 1), (1, 0, 1), (0, 1, 1), (1, 1, 1)$, oriented outward.

Solution.

$$\begin{aligned}\operatorname{div}\mathbf{F} &= \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(-3xy) + \frac{\partial}{\partial z}(xz^2) \\ &= 2 - 3x + 2xz\end{aligned}$$

The integration region is a unit cube.

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iiint_E \operatorname{div}\mathbf{F}\, dV \\ &= \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} (2 - 3x + 2xz)\, dz\, dy\, dx \\ &= \int_{x=0}^{1}\int_{y=0}^{1} \left[2z - 3xz + xz^2\right]_{z=0}^{1} dy\, dx \\ &= \int_{x=0}^{1}\int_{y=0}^{1} [2 - 3x + x]\; dy\, dx \\ &= \int_{x=0}^{1} [2 - 3x + x]\; dx \\ &= \int_{x=0}^{1} [2 - 2x]\; dx \\ &= \left[2x - x^2\right]_{x=0}^{1} \\ &= 2 - 1 \\ &= 1\end{aligned}$$